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Chapter 19 *The time value of money*

Proof of incremental payments formula

Proof that

$$S = A_0\left(1+\frac{r}{100}\right)^t + \frac{x\left(1+\frac{r}{100}\right)^t - x}{\frac{r}{100}}$$

where S is the sum at the end of t years, A_0 is the initial investment and x is the amount added at the end of each year.

The value of the investment at the end of the first year is

$$S_1 = A_0\left(1+\frac{r}{100}\right)^1 + x$$

The value at the end of the second year is

$$S_2 = \left[A_0\left(1+\frac{r}{100}\right)^1 + x\right]\left(1+\frac{r}{100}\right)^1 + x$$
$$= A_0\left(1+\frac{r}{100}\right)^2 + x\left(1+\frac{r}{100}\right)^1 + x$$

If we continue in this way, it can be shown that the value after t years is

$$S = A_0\left(1+\frac{r}{100}\right)^t + x\left(1+\frac{r}{100}\right)^{t-1} + x\left(1+\frac{r}{100}\right)^{t-2} + \cdots + x$$

This can be simplified using the summation formula for a geometric progression to give

$$S = A_0\left(1+\frac{r}{100}\right)^t + x\left[\frac{1-\left(1+\frac{r}{100}\right)^t}{1-\left(1+\frac{r}{100}\right)}\right]$$
$$= A_0\left(1+\frac{r}{100}\right)^t + x\left[\frac{1-\left(1+\frac{r}{100}\right)^t}{\frac{-r}{100}}\right]$$
$$= A_0\left(1+\frac{r}{100}\right)^t + \frac{x\left(1+\frac{r}{100}\right)^t - x}{\frac{r}{100}}$$